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A perturbational approach to magneto-thermal problems of a deformed sphere levitated in a magnetic field

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Abstract. This paper presents a perturbation method for the solution of the electromagnetic and thermal problems of a deformed sphere levitated in an alternating magnetic field. The analytical solutions of the electromagnetic field distribution, the Joule heat generation, the magnetic lifting force and the temperature field are obtained based on a linear perturbation theory. The Maxwell equations are first simplified in terms of the vector potential and then solved by the method of separation of variables. The time-averaged Joule-heat source is calculated and coupled to the Fourier heat-conduction equation. The coupled equation is solved for temperature distributions within the deformed sphere by a combined approach of series expansion and variation of parameters. Both asymptotic and numerical analyses are provided. The total power absorption and temperature field for both single and multiple coils are also discussed.

Key words: Magnetic livetation; magneto-thermal phenomena; perturbation method; Joule heating; Legendre functions

I. Introduction

Magnetic levitation is a major process for achieving a containerless environment for the purpose of metal refining or purification. The basic idea for magnetic levitation may be briefly described as follows. When an electrically conducting sample, such as metal, is placed in an alternating magnetic field, currents will be induced in the sample. These induced currents will generate a magnetic field which will impress upon the applied magnetic field. The induced currents interact with the total (*i.e.*, the imposed and the induced) magnetic field to produce electromagnetic forces in the sample. If the applied current is strong enough, the electromagnetic forces can counterbalance the gravitational force to levitate the sample in space. Also, the induced currents will interact with themselves to generate a Joule-heating effect which heats up or eventually melts the sample being levitated. A distinct advantage of the process is that levitation creates a containerless environment and prevents the sample from being contaminated by any container-related impurities, thereby providing a unique way to obtain extra high-purity materials. Recent research work suggests [1–3] that aside from advantages of containerless purification, the magnetic-levitation technique also can be applied to achieve a significant amount of undercooling, measured by the difference between the actual temperature and the melting point of a melt, which would otherwise be difficult to accomplish by the container-based technology. The process is also being explored as a means for directly measuring the thermal and physical properties of conducting materials and undercooled melts.

Because of these unique advantages, magnetic levitation has received a great deal of attention within the research community. Okress, *et al.* [4] are perhaps the first to have

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started laboratory research on the subject. Recent work has been on developing a better understanding of both the electrodynamic and transport phenomena occurring in the magnetic-levitation process [5–13]. Analytical studies published so far have been concerned with the behavior of a perfect sphere, primarily because sphericity offers a simple and solvable system. These solutions represent basically zero-order approximations, since in reality a sample under levitation is in the molten stage and cannot remain spherical. For these molten droplets, the final equilibrium free-surface shape is determined by a detailed balance of surface tension, electromagnetic forces, gravitational forces and hydrodynamic forces along the surface, and numerical algorithms have been well established for the purpose [9, 10]. There has been much interest in obtaining information on the temperature distribution in a deformed sample levitated in an electromagnetic magnetic field. Such information is of critical value in designing a magnetic-levitation system as well as in interpreting experimental measurements taken from a sample levitated magnetically. Despite this importance, analyses of thermal behavior of a molten droplet levitated magnetically do not appear to have been carried out.

In this paper, we present a perturbational approach to the magnetothermal problems related to an aspherical sample which is levitated magnetically. We will consider a slightly deformed sphere with axial symmetry and solve for the electromagnetic-field distribution, the Joule heating, and temperature distribution in magnetic-levitation processes. The electromagnetic field induced in the spheroid is obtained analytically via the magnetic vector potential by a perturbation method up to linear-order accuracy. The Joule heating is derived from the known induced current-density distribution. The heating contributes to the temperature distribution in the sample as a source term. The solution of the temperature distribution will also be obtained analytically, again by a perturbation method, up to linear accuracy. These solutions are presented first for a single coil configuration, which offers an ideal system for analytical approaches. The extension of the solutions to treat a multiple-coil configuration is also discussed. With the derived formulae, the time-averaged power absorption, an important quantity characterizing the thermal aspects of a levitation system, is discussed. Some asymptotic analyses, based on the perturbational solutions, are also presented and numerical results are given.

Some necessary assumptions and simplifications have been made to render the problem analytically workable. First, we have assumed that the sample suspended in the magnetic field satisfies suitable stability conditions and thus oscillations or rotation with respect to the vertical axis do not occur; this is the prerequisite for the system to be of axial symmetry. We have relaxed the previous requirement of perfect sphericity for the sample, but will attach the condition that the deformation is small and is within the realm of perturbation analyses. This assumption is thought to be reasonable for a levitation system in which a sample is only slightly deformed by carefully arranged coil configurations. In seeking the solution of the temperature field, we have assumed that heat transfer in the sphere is by conduction. This assumption can be true for a sample started with an imperfect spherical shape and heated before it gets melted; but it can be a severe assumption, especially when a steady-state temperature distribution is sought for the liquid sphere, as in reality thermal convection also takes place in the liquid sample and helps to even the temperature difference. Because of this, the predicted thermal field is only partially correct for a levitated liquid sample. Nonetheless, our perturbational solutions, which are possible with these necessary assumptions, should be of fundamental value in assessing the magnetothermal behavior of a levitation system, and also provide useful guidelines for magnetic-levitation process design and development.

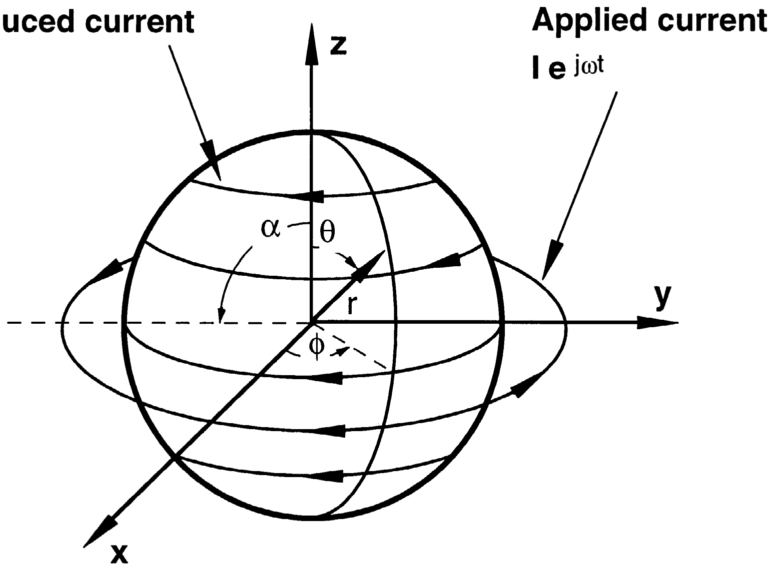

Figure 1. Schematic representation of a magnetic-levitation system. The spherical coordinates used in the analyses are also shown.

II. Problem formulation

Consider the axisymmetric system to be investigated as illustrated in Figure 1. In practice, two types of coils are used: one for the purpose of levitation and the other for heating. The levitation coils are so designed and placed that the specimen can be levitated without proneness to instability: any perturbation from the equilibrium position will be damped away. Usually, these coils provide the thermal energy needed to keep the sample slightly above the melting point after the process reaches its steady state. The heating coils, on the other hand, are used primarily to heat and melt the sample, for the purpose of which a higher current with a higher frequency is used. The whole levitation system is normally sealed in a vacuum vessel, or immersed in an inert gas, such as helium, environment, to prevent the sample from oxidizing at high temperatures. To understand the magnetothermal phenomena involved in the process, the electromagnetic field and its interaction with the temperature field must be resolved.

(a) The electromagnetic field

Our theoretical analysis of the above system starts with the electromagnetic field, which, in general, is described by the Maxwell equations [14,15]

$$\nabla \cdot \mathbf{D} = \rho_e, \tag{1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{4}$$

where ρ_e is the free-charge density, **D** the displacement current, **E** the electric field, **B** the magnetic field, **H** the magnetic intensity and **J** the current density.

Clearly a direct solution of these equations would be a formidable task and yet some simplifications may be made so as to facilitate a solution for our particular system. For normal levitation conditions, the quasi-steady state or near-field approximations are quite valid. As the frequency used in the levitation device is relatively low compared with the mean collision frequency of electron gas in the metal, there will be no electric charge separation and hence the alteration in conductivity may well be neglected. Also, as the levitation apparatus is normally much smaller than the electromagnetic wave length generated by the applied current, the displacement current can be safely ignored [14]. For a sinusoidal field like the present one, the field quantities may be conveniently represented in phasor notation. For example, the magnetic induction may be separated into two parts and written in the form of

$$\mathbf{B} = \hat{\mathbf{B}} e^{j\omega t}, \tag{5}$$

with $\hat{\mathbf{B}}(\mathbf{r})$ being a complex-variable amplitude having only spatial dependence, ω the angular frequency of the imposed field and $j = \sqrt{-1}$. Other field quantities may be written similarly. Incorporating these simplifications, then making use of the magnetic vector potential, defined as,

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{6}$$

and finally with some vector manipulations, we can simplify the Maxwell equations and express the result in terms of the magnetic vector potential, *viz*.,

$$\nabla^2 \hat{\mathbf{A}} - j\omega\mu\sigma\hat{\mathbf{A}} = 0, \tag{7}$$

where we have made use of the Coulomb gauge [14, 15], and σ and μ are the electrical conductivity and magnetic permeability, respectively.

The condition that the system under consideration possesses axisymmetry requires that the r- and θ- components of the magnetic vector potential be identically zero. Thus, only the azimuthal component of the magnetic vector potential, A_ϕ, survives. We can then write the governing equation for the magnetic field in the levitation system in terms of A_ϕ,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\right) - \frac{A_\phi}{r^2\sin^2\theta} = k^2 A_\phi, \tag{8}$$

where k is a parameter for the levitation system,

$$k^2 = j\omega\mu\sigma. \tag{9}$$

For the sake of brevity, we have dropped the caret on A_ϕ in Equation (2.14) and will henceforth do the same for the electromagnetic-field quantities, unless otherwise indicated. It is noted here that the above equation applies to both inside and outside the conducting sphere. When applied to the outside, it becomes homogeneous as k reduces to zero.

The boundary conditions are that the tangential magnetic field and the normal magnetic field, involving no magnetization, must be continuous along the interface between the air and the sphere, so as to give rise to the following relationships on the surface of the deformed sphere [15,16]:

$$A_{\phi i} = A_{\phi 0}; r \in \partial\Omega, \tag{10}$$

$$\frac{\partial A_{\phi i}}{\partial n} = \frac{\partial A_{\phi 0}}{\partial n}; r \in \partial\Omega, \tag{11}$$

where the subscripts refer to the inside and outside fields, respectively.

(b) The temperature field

For the conducting sample, the heating results primarily from the Joule effect. The final temperature distribution within the sample is a combined result of Joule heating and heat loss to the environment. When the Joule heating is balanced by the heat loss, a steady-state condition occurs. In this paper, we consider the steady state only. Within the framework of the assumptions given in Section I, the energy-balance equation for the steady-state condition may be written thus [17],

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{Q(r,\theta)}{K}=0, \qquad (12)$$

where T is the temperature, Q the Joule-heating source, and K the thermal conductivity. Clearly, the Joule-heating term represents the coupling between the electromagnetic and temperature fields.

The cooling of the sample in a levitation system is either from radiation to the chamber walls or from the convective loss to the inert gas (He or Ar) or both. To simplify the nonlinear effects associated with these boundary conditions, the cooling is assumed to follow the linear convective law on the deformed surface,

$$-K\frac{\partial T}{\partial n}=H_{\mathit{eff}}T, \qquad (13)$$

where H_{eff} is the effective heat-transfer coefficient and the temperature is measured from $T_\infty = 298$, which is the environment temperature. In writing the above equation, we have assumed that the levitation system is being operated under vacuum and thus the heat loss to the environment comes primarily from thermal radiation. For a sample immersed in a stream of He gas, an additional convective coefficient would have to be added. For either case, H_{eff} is assumed to be constant and is calculated as an averaged value [11].

III. The perturbation method

We seek a solution to the above equations describing the electrodynamic and thermal phenomena in a deformed sphere via a regular perturbational approach [18]. By this method, for a deformed sphere whose boundary is prescribed by a series expansion around the corresponding sphere

$$r=a\left(1+\varepsilon\sum_{i=1}f^i(\cos\theta)\right), \qquad (14)$$

the solution of a field variable may be expanded in terms of the regular boundary perturbation parameter, ε, or specifically,

$$u(\mathbf{r},\varepsilon)=u^0(\mathbf{r})+\varepsilon u^1(\mathbf{r})+\varepsilon^2u^2(\mathbf{r})+\cdots, \qquad (15)$$

where a is the radius of the sphere and the terms associated with ε represent higher-order approximations.

As the boundary is irregular, the field variable of any order must be expanded in the boundary perturbational form. For a slight departure from a sphere of radius a, the field variables of any order approximation may be expanded as a Taylor series around a,

$$u^i(r,\varepsilon) = u^i(a) + \sum_{j=1} \frac{(r-a)^j}{j!} \frac{\partial^j u^i(r)}{\partial r^j}\bigg|_{r=a}; \quad r \in \partial\Omega. \tag{16}$$

Since $r - a = a\varepsilon \sum_{i=1} f^i(\theta)$ on $\partial\Omega$, the above equation then takes the form of

$$u^i(r,\varepsilon) = u^i(a) + \sum_{j=1} \frac{\varepsilon^j a^j (\sum_{m=1} f^m_{(\theta)})^j}{j!} \frac{\partial^j u^i(r)}{\partial r^j}\bigg|_{r=a}; \quad r \in \partial\Omega. \tag{17}$$

Similarly, the nth-order derivative of the field variables of any order approximation can be expanded in a Taylor series around a sphere of radius a along the boundary,

$$\frac{\partial^n u^i(r,\varepsilon)}{\partial r^n} = \frac{\partial^n u^i}{\partial r^n}\bigg|_{r=a} + \sum_{j=1} \frac{\varepsilon^j a^j (\sum_{m=1} f^m_{(\theta)})^j}{j!} \frac{\partial^{j+n} u^i(r)}{\partial r^{j+n}}\bigg|_{r=a}; \quad r \in \partial\Omega. \tag{18}$$

To obtain the perturbational form of the boundary conditions involving normal derivatives, we further need an expansion for the normal derivatives to the deformed surface,

$$\begin{aligned}
\frac{\partial u}{\partial n} &= n \cdot \nabla u \\
&= \left[1 + \left(\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]^{-1/2} \left(\hat{e}_r - \frac{\hat{e}_\theta}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) \cdot \left(\hat{e}_r \frac{\partial u}{\partial r} + \frac{\hat{e}_r}{r}\frac{\partial u}{\partial \theta}\right)\bigg|_{r=a} \\
&= \left[1 + \left(\frac{\varepsilon \sum_{j=1} \mathrm{d}f^j(\cos\theta)/\mathrm{d}\theta}{\left(1 + \varepsilon \sum_{j=1} f^j(\cos\theta)\right)}\right)^2\right]^{-1/2} \\
&\quad \times \left[\frac{\partial u}{\partial r} - \frac{\varepsilon \sum_{j=1} \mathrm{d}f^j(\cos\theta)/\mathrm{d}\theta}{a\left(1 + \varepsilon \sum_{j=1} f^j(\cos\theta)\right)^2} \frac{\partial u}{\partial \theta}\right] \\
&\approx \frac{\partial u}{\partial r} - \frac{\varepsilon}{a} \sum_{j=1} \frac{\mathrm{d}f^j(\cos\theta)}{\mathrm{d}\theta} \frac{\partial u}{\partial \theta}; \quad r \in \partial\Omega.
\end{aligned} \tag{19}$$

The above approximation for the normal derivative is accurate up to the order of ε. It is also obvious that, if u and $\partial u/\partial n$ are both continuous at $r \in \partial\Omega$, then $\partial u/\partial r$ is continuous across the surface of the deformed sphere.

In theory the above method can be applied to obtain the perturbational approximations of any order in the same fashion. In practice, however, calculations are very complex when second-order or higher approximations are considered. Thus, we will obtain the perturbational solutions only up to linear order. To simplify our analyses, we further assume that the deformed surface of the sphere is described by the following function

$$r(\theta) = a(1 + \varepsilon P_2(\cos\theta)) \tag{20}$$

on the basis of which the above boundary perturbation terms can be simplified and our perturbation solutions of the electromagnetic field and temperature distribution within the levitated sample will be obtained.

IV. The solution of the electromagnetic field

In accordance with linear perturbation, the magnetic vector potential that is sought will assume the following form,

$$A_\phi = A_\phi^0 + \varepsilon A_\phi^1. \tag{21}$$

Substituting this in Equation (8) and collecting terms of the same order, we obtain the following two equations for the vector potential,

$$\varepsilon^0 : \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi^0}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial_\theta}\left(\sin\theta\frac{\partial A_\phi^0}{\partial\theta}\right) - \frac{A_\phi^0}{r^2\sin^2\theta} = k^2A_\phi^0, \tag{22}$$

$$\varepsilon^1 : \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi^1}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial_\theta}\left(\sin\theta\frac{\partial A_\phi^1}{\partial\theta}\right) - \frac{A_\phi^1}{r^2\sin^2\theta} = k^2A_\phi^1, \tag{23}$$

Applying the boundary perturbation expansions as described in the last section, subject to the necessary simplifications resulting from the boundary shape prescribed by Equation (14), we have up to the second order,

$$A_\phi = A_\phi^0|_{r=a} + a\varepsilon P_2(\cos\theta)\left.\frac{\partial A_\phi^0}{\partial r}\right|_{r=a} + \varepsilon A_\phi^1|_{r=a} + a\varepsilon^2P_2(\cos\theta)\left.\frac{\partial A_\phi^1}{\partial r}\right|_{r=a}. \tag{24}$$

If we substitute this in Equation (10), we find two conditions, each corresponding to its specific order,

$$\varepsilon^0 : A_{\phi_i}^0|_{r=a} = A_{\phi_0}^0|_{r=a}, \tag{25}$$

$$\varepsilon^1 : A_{\phi_i}^1|_{r=a} = A_{\phi_0}^1|_{r=a}. \tag{26}$$

Clearly, the condition that the vector potential is continuous across the boundary remains true up to linear order.

Similarly, we may apply Equations(17), (18), (22) and (23) to derive the boundary conditions governing the normal derivatives of the first and second orders across the interface between the sphere and the environment. Applying some algebra, we may show that

$$\varepsilon^0 : \frac{\partial A_{\phi i}^0}{\partial r} = \left.\frac{\partial A_{\phi 0}^0}{\partial r}\right|_{r=a}, \tag{27}$$

$$\varepsilon^1 : \left.\frac{\partial A_{\phi i}^1}{\partial r}\right|_{r=a} = \left.\frac{\partial A_{\phi 0}^1}{\partial r}\right|_{r=a} - aP_2(\cos\theta)k^2A_{\phi i}^0\,. \tag{28}$$

The first boundary condition represents the continuity of the normal derivative across the boundary for a perfect sphere, which is consistent with the expression given in Equation (11). The boundary condition for the linear terms shows that continuity cannot be maintained at higher orders. The normal derivatives change across the boundary, as indicated by the second term on the right-hand side of the equation, which represents the contribution of the zero-order terms.

It is noted here that in arriving at Equation (26) we already made use of Equation (27). Also, in Equation(28), the last term which represents the effect of the zero-order solution, takes a remarkably simple form for this particular problem. In general, it would involve the higher-order derivative of the zero-order solution.

The set of Equations (22), (25) and (27) represents the zero-order approximation or solution for a perfect sphere. The solution was obtained by Li [11] and is re-written below for convenience,

$$\left.\begin{aligned} A^0_{\phi i} &= \frac{\mu I \sin\alpha}{2}\left(\frac{a}{r}\right)^{1/2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left(\frac{a}{r_0}\right)^n \frac{P_n^1(\cos\alpha)P_n^1(\cos\theta)I_{n+1/2}(kr)}{kaI_{n-1/2}(ka)}; \quad r \leqslant a,\\ A^0_{\phi 0} &= \frac{\mu I \sin\alpha}{2}\sum_{n=1}^{\infty}\frac{P_n^1(\cos\alpha)P_n^1(\cos\theta)}{n(n+1)}\left(\frac{r_0 r_<^n}{r_>^{n+1}} - \frac{I_{n+3/2}(ka)}{I_{n-1/2}(ka)}\left(\frac{a}{r}\right)^{n+1}\left(\frac{a}{r_0}\right)^n\right); \quad r > a, \end{aligned}\right\} \tag{29}$$

where $P_n^1(x)$ is the Legendre function, and $I_{n+1/2}(x)$ the modified Bessel function of the first kind and $r_<(r_>)$ is the smaller (larger) of r_o and r.

Equation (23), along with the boundary conditions expressed by Equations (26) and (28), constitutes the linear-order approximation and is now solved. The equation set is separable and we will take the same approach as detailed in [11]. By that approach, we first treat the whole space as if it were a homogeneous medium with the origin of the coordinate system located at the center of the sphere. The solution is then simplified by setting the system parameter k equal to zero for the outside of the deformed sphere. Thus, the solution for both the outside and inside of the sphere, after separation of variables [19], has the following form,

$$A^1_{\phi i} = \sum_{n=1}^{\infty} C_n r^{-1/2} I_{n+1/2}(kr) P_n^1(\cos\theta), \tag{30}$$

$$A^1_{\phi 0} = \sum_{n=1}^{\infty} C_n r^{-1/2} K_{n+1/2}(kr) P_n^1(\cos\theta). \tag{31}$$

In a normal levitation operation, the environment is either a vacuum or air and thus the conductivity is virtually zero. Making use of the asymptotic behavior of the modified Bessel functions of the second kind as the argument of the function, x, tends to zero [20],

$$K_{n+1/2}(x) = \begin{cases} -\left[\log\left(\dfrac{x}{2}\right) + 0.5772\cdots\right], & n = -1/2,\\ \dfrac{\Gamma(n+1/2)}{2}\left(\dfrac{2}{x}\right)^{n+1/2}, & n \neq -1/2, \end{cases} \tag{32}$$

we can show that Equation (31) reduces to

$$A^1_{\phi 0} = \sum_{n=1}^{\infty} D_n P_n^1(\cos\theta)/r^{n+1}, \tag{33}$$

which could have been obtained from Equation (23) directly by the method of separation of variables, when k is set equal to zero. It is remarked here that the use of Equations (30) and (31) represents a unified approach and is more convenient, as both the outside and inside solutions can be readily deduced from a single separation-of-variables procedure.

The constants C_n^1 and D_n^1 in Equations (30) and (33) can now be determined from the two first-order boundary conditions. Substituting Equations (30) and (33) in Equations (26) and (28), respectively, we have

$$C_n^1 I_{n+1/2}(ka)a^{-1/2} = D_n^1 a^{-n-1}, \tag{34}$$

$$\frac{C_n^1 I_{n-1/2}(ka) P_n^1(\cos\theta)}{a^{1/2}} = -P_2(\cos\theta)\,\frac{\mu I \sin\alpha}{2}\,\frac{2n+1}{n(n+1)} \times \left(\frac{a}{r_0}\right)^n \frac{P_n^1(\cos\alpha)P_n^1(\cos\theta)I_{n+1/2}(ka)}{I_{n-1/2}(ka)}\,. \tag{35}$$

To obtain the expression for C_n^1, the right-hand side must also be expanded in the form of Legendre functions. While the normal procedures may be followed to get the expansion series, a more simplified approach would be to use the recursive relationship between the Legendre functions of different orders. For associated Legendre functions of varying degrees, we have the following recurrence relation [20]:

$$(n-m+1)P_{n+1}^m(x) - (2n+1)xP_n^m(x) + (n+m)P_{n-1}^m = 0. \tag{36}$$

Also, with the definition of$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ and by repeated use of Equation (46), one can get the following expression for the expansion of $P_2(\cos\theta)P_n^1(\cos\theta)$ in terms of $P_n^1(\cos\theta)$,

$$\begin{aligned} P_2(\cos\theta)P_n^1(\cos\theta) &= \frac{3}{2}\cos^2\theta P_n^1(\cos\theta) - \frac{1}{2}P_n^1(\cos\theta) \\ &= \frac{3n(n+1)}{2(2n+1)(2n+3)}\,P_{n+2}^1(\cos\theta) + \frac{n^2+n-3}{(2n+3)(2n-1)}\,P_n^1(\cos\theta) \\ &\quad + \frac{3n(n+1)}{2(2n+1)(2n-1)}\,P_{n-2}^1(\cos\theta)\,. \end{aligned} \tag{37}$$

Using this relation, we can obtain C_n^1 and D_n^1. Combining with the zero-order solution, we have the perturbational form of the vector potential with a linear accuracy in ε for both the inside and outside of the sphere, *viz*.,

$$\begin{aligned} A_{\phi i} &= A_{\phi i}^0 + \varepsilon A_{\phi i}^1 \\ &= \frac{\mu I \sin\alpha}{2}\left(\frac{a}{r}\right)^{1/2} \sum_{n=1}^{\infty} \frac{P_n^1(\cos\theta)I_{n+1/2}(kr)}{I_{n-1/2}(ka)} \left\{\frac{2n+1}{n(n+1)}\,\frac{P_n^1(\cos\alpha)}{ka}\left(\frac{a}{r_0}\right)^n + \varepsilon E_n\right\}, \end{aligned} \tag{38}$$

$$\begin{aligned} A_{\phi 0} &= A_{\phi 0}^0 + \varepsilon A_{\phi 0}^1 \\ &= \frac{\mu I \sin\alpha}{2} \sum_{n=1}^{\infty} P_n^1(\cos\theta)\left[\frac{P_n^1(\cos\alpha)}{n(n+1)}\left(\frac{r_0 r_<^n}{r_>^{n+1}} - \frac{I_{n+3/2}(ka)}{I_{n-1/2}(ka)}\left(\frac{a}{r}\right)^{n+1}\left(\frac{a}{r_0}\right)^n\right)\right. \\ &\qquad \left. + \varepsilon E_n\,\frac{I_{n+1/2}(ka)}{I_{n-1/2}(ka)}\left(\frac{a}{r}\right)^{n+1}\right], \end{aligned} \tag{39}$$

where the coefficient E_n is evaluated according to

$$E_n + B_{n-2}\,\frac{3(n-1)(n-2)}{2(2n-1)(2n-3)} + B_n\,\frac{n^2+n-3}{(2n+3)(2n-1)} + B_{n+2}\,\frac{3(n+2)(n+3)}{2(2n+5)(2n+3)}\,, \tag{40}$$

with B_n defined by

$$B_n = -\frac{2n+1}{n(n+1)} P_n^1(\cos\alpha) \left(\frac{a}{r_0}\right)^n \frac{I_{n+1/2}(ka)}{I_{n-1/2}(ka)} .$$

With the vector potential known, we may calculate other electromagnetic-field variables. One of the quantities which are of direct interest, given the objective of this paper, is the time-averaged Joule heating within the deformed sphere, and may be calculated as follows:

$$\begin{aligned} Q(r,\theta) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{Re}(\mathbf{J}e^{j\omega t}) \cdot \mathrm{Re}(\mathbf{J}e^{j\omega t})}{\sigma}\, \mathrm{d}t \\ &= \frac{J_\phi^0 \cdot J_\phi^{0*}}{2\sigma} + \frac{\varepsilon}{\sigma} \mathrm{Re}(J_\phi^0 \cdot J_\phi^{1*}) , \end{aligned} \tag{41}$$

where $J_\phi = -j\omega\sigma A_{\phi i}$ is the induced eddy-current density and also the Joule-heating source has been linearized with respect to ε so as to be consistent with the linear-order analyses.

V. The solution of the temperature field

The procedure for the perturbation solution of the temperature field is similar. For the linear approximation considered here, the temperature distribution takes the following form,

$$T(r,\varepsilon) + T^\infty(r) + \varepsilon T^1(r) . \tag{42}$$

Upon substitution in Equation (12) and rearranging, we can write the equations describing the zero and first-order approximations for the temperature-field distribution in the deformed sphere explicitly as follows:

$$\varepsilon^0 : \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T^0}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial T^0}{\partial\theta} \right) + \frac{J_\phi^0 \cdot J_\phi^{0*}}{2\sigma K} = 0 , \tag{43}$$

$$\varepsilon^1 : \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T^1}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial T^1}{\partial\theta} \right) + \frac{\mathrm{Re}(J_\phi^0 \cdot J_\phi^{1*})}{\sigma K} = 0 . \tag{44}$$

The convective boundary conditions of the corresponding order can also be obtained when we follow the same boundary perturbation procedure, whence we have,

$$\frac{\partial T}{\partial n} + hT = \frac{\partial T}{\partial r} + \frac{\varepsilon}{a} P_2^1(\cos\theta) \frac{\partial T}{\partial\theta} + hT = 0 . \tag{45}$$

Therefore, we have for the zeroth and first-order conditions,

$$\varepsilon^0 : \frac{\partial T^0}{\partial r} + hT^0 = 0 , \tag{46}$$

$$\varepsilon^1 : \frac{\partial T^1}{\partial r} + hT^1 = -aP_2(\cos\theta) \left(\frac{\partial^2 T^0}{\partial r^2} + h \frac{\partial T^0}{\partial r} \right) - \frac{P_2^1(\cos\theta)}{a} \frac{\partial T^0}{\partial\theta} , \tag{47}$$

where $h = H_{\mathit{eff}}/K$. Note here that the temperature field does not have the fortunate property of the electromagnetic field and the higher derivatives must be spelt out by following through the straightforward, yet laborious derivations.

As with the magnetic vector potential, the zero-order solution for the problem, as defined by Equations (43) and (46), has already been obtained before [11, 21],

$$T^0(r,\theta) = \frac{\mu\omega I^2 \sin^2\alpha}{16K} \sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{2m+1}{m(m+1)}\,\frac{2n+1}{n(n+1)} \left(\frac{a}{r_0}\right)^{n+m} P_m^1(\cos\alpha)P_n^1(\cos\alpha)P_l(\cos\theta)I_{lmn}(r)P_{lmn}\,. \tag{48}$$

In the above equation, P_{lmn} is an integral of a triple product of associated Legendre functions of mixed order and was evaluated elsewhere [11],

$$\begin{aligned} P_{lmn} &= \int_0^1 P_l(\xi)P_m^1(\xi)P_n^1(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2}\left(-l(l+1)+m(m+1)+n(n+1)\right)\int_{-1}^{1} P_l(\xi)P_m(\xi)P_n(\xi)\,\mathrm{d}\xi \\ &= \left(-l(l+1)+m(m+1)+n(n+1)\right)\frac{(2s)!!}{(2s+1)!!} \\ &\quad\times\frac{(2(s-l)-1)!!(2(s-m)-1)!!(2(s-n)-1)!!}{(2(s-l))!!(2(s-m))!!(2(s-n)!!}\,, \end{aligned} \tag{49}$$

$$2s = l+m+n = \textit{Even}, \qquad l \geqslant 0; 0 \leqslant m-l \leqslant n \leqslant m+l$$

and $I_{lmn}(r)$ is a function of r

$$\begin{aligned} I_{lmn}(r) = \frac{1}{a}\Bigg[&\int_0^r \left(\frac{r'}{r}\right)^{l+1} \mathrm{Re}[R_{mn}(r')]\,\mathrm{d}r' + \int_r^a \left(\frac{r}{r'}\right)^{l} \mathrm{Re}[R_{mn}(r')]\,\mathrm{d}r' \\ &+ \left(\frac{r}{a}\right)^{l} \frac{(l+1-ha)}{(ha+l)} \int_0^a \left(\frac{r'}{a}\right)^{l+1} \mathrm{Re}[R_{mn}(r')]\,\mathrm{d}r'\Bigg]\,, \end{aligned} \tag{50}$$

where $R_{mn}(r)$ involves a product of modified Bessel functions of different orders

$$R_{mn}(r) = \frac{I_{m+1/2}(kr)}{I_{m-1/2}(ka)}\,\frac{I_{n+1/2}(k^*r)}{I_{n-1/2}(k^*a)}\,. \tag{51}$$

We now consider the solution of the linear order for the set of Equations (44) and (47). Inspection of the equation set indicates that the solution is separable,

$$T(r,\theta) = \sum_{l=0} R_l(r)P_l(\cos\theta)\,. \tag{52}$$

To facilitate the solution, we also expand the Joule-heating source term in terms of Legendre functions

$$\frac{\mathrm{Re}(J_\phi^0 \cdot J_\phi^{1*})}{\sigma K} = \frac{\mu\omega I^2\sin^2\alpha}{16K}\left(\frac{1}{r}\right)\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{lmn}(r)\,, \tag{53}$$

where $F_{lmn}(r)$ is a function of r

$$F_{lmn}(r) = P_{lmn}P_m^1(\cos\alpha)\,\frac{4(2m+1)}{m(m+1)}\left(\frac{a}{r_0}\right)^m \mathrm{Re}[k^* E_n^* R_{mn}(r)]\,. \tag{54}$$

Upon substituting the above two equations in Equation (44) and rearranging, we have an inhomogeneous ordinary differential equation for $R(r)$

$$\frac{\mathrm{d}^2R_l(r)}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}R_l(r)}{\mathrm{d}r}-\frac{l(l+1)}{r^2}R_l(r)+\frac{\mu\omega I^2\sin^2\alpha}{16K}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(2l+1)F_{lmn}(r)}{r}=0\,. \tag{55}$$

Equation (55) may be solved readily along with Equation (47). After some algebraic manipulations, we have the first-order approximation to the temperature distribution

$$\begin{aligned}T^1(r,\theta) \;=\; &\frac{\mu\omega I^2\sin^2\alpha}{16K}\sum_{l=0}^{\infty}P_l(\cos\theta)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left[\int_0^r\left(\frac{r'}{r}\right)^{l+1}F_{lmn}(r')\,\mathrm{d}r'\right.\right.\\ &+\left.\int_r^a\left(\frac{r}{r'}\right)^{l}F_{lmn}(r')\,\mathrm{d}r'\right]+\left(\frac{r}{a}\right)^{l}\left[\frac{(l+1-ha)}{l+ha}\int_0^a\left(\frac{r'}{a}\right)^{l+1}F_{lmn}(r')\,\mathrm{d}r'\right.\\ &-\left.\left.\frac{g_{lmn}}{l+ha}\,\frac{2m+1}{m(m+1)}\,\frac{2n+1}{n(n+1)}\left(\frac{a}{r_0}\right)^{m+n}P_m^1(\cos\alpha)P_n^1(\cos\alpha)\right]\right\}\,.\end{aligned} \tag{56}$$

In the above equation g_{lmn} is a constant defined as

$$\begin{aligned}g_{lmn} \;=\; &\{a^2[S1_{l-2}P_{(l-2)mn}D_{(l-2)mn}+S2_lP_{lmn}D_{lmn}+S3_{l+2}P_{(l+2)mn}D_{(l+2)mn}]\\ &+[T1_{l-2}P_{(l-2)mn}I_{(l-2)mn}(a)+T2_lP_{lmn}I_{(l-2)mn}(a)\\ &+S3_{l+2}P_{(l+2)mn}I_{(l+2)mn}(a)]\}\,,\end{aligned} \tag{57}$$

where

$$S1_l=\frac{3(l+1)(l+2)}{2(2l+1)(2l+3)};\quad S2_l=\frac{l(l+1)}{(2l-1)(2l+3)};\quad S3_l=\frac{3l(l-1)}{2(2l+1)(2l-1)}$$

$$T1_l=\frac{3l(l+1)(l+2)}{(2l+1)(2l+3)};\quad T2_l=\frac{3l(l+1)}{(2l-1)(2l+3)};\quad T3_l=\frac{3l(l+1)(l-1)}{(2l+1)(2l-1)}$$

$$\begin{aligned}D_{lmn}+&\frac{(l+1)(l+2-ha)(ha+l)+l^3-l(1-ha)^2}{(ha+l)}\\ &\times\int_0^a\left(\frac{r'}{a}\right)\,\mathrm{Re}[R_{mn}(r'/a)]\,\mathrm{d}(r'/a)-(2l+1)\,\mathrm{Re}[R_{mn}(a)]\,.\end{aligned}$$

The final temperature solution thus is a linear combination of the zeroth and linear order approximations.

VI. Results and discussion

The above formulae may be used to derive some useful quantities for the magnetic levitation of the deformed sphere. Of particular importance to the thermal aspects of the levitation system is the total power absorbed by the sphere, which may be obtained when we integrate the analytical solutions.

(a) Total power absorption

The power absorption represents the total heat generated within the conducting body and we may calculate this by integrating the distributed Joule-heating over the entire deformed sphere,

$$\begin{aligned} Q_{tot} &= \int_\Omega \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{Re}(\mathbf{J}e^{j\omega t}) \cdot \mathrm{Re}(\mathbf{J}e^{j\omega t})}{\sigma} \, \mathrm{d}t\mathrm{d}\Omega \,, \\ &= \int_\Omega \left[\frac{J_\phi^0 \cdot J_\phi^{0*}}{2\sigma} + \frac{\varepsilon}{\sigma} \mathrm{Re}(J_\phi^0 \cdot J_\phi^{1*}) \right] \, \mathrm{d}\Omega \,. \end{aligned} \tag{58}$$

In line with linear perturbation theory, we can also split Equation (58) into two terms, one involving a double integral and the other a single integral,

$$\begin{aligned} Q_{tot} &= 2\pi \int_{-1}^{1} \int_0^{a+a\varepsilon P_2(\cos\theta)} Q^0(r,\cos\theta) r^2 \, \mathrm{d}r \, \mathrm{d}\cos\theta \\ &\quad +\varepsilon 2\pi \int_{-1}^{1} \int_0^{a+a\varepsilon P_2(\cos\theta)} Q^1(r,\cos\theta) r^2 \, \mathrm{d}r \, \mathrm{d}\cos\theta \\ &= 2\pi \int_{-1}^{1} \int_0^{a} Q(r,\cos\theta) r^2 \, \mathrm{d}r \, \mathrm{d}\cos\theta \\ &\quad +\varepsilon 2\pi a^3 \int_{-1}^{1} P_2(\cos\theta) Q^0(a,\cos\theta) \, \mathrm{d}\cos\theta \,. \end{aligned} \tag{59}$$

Substituting the expressions for J_ϕ^0 and J_ϕ^1 in the first term of the above equation we arrive at a complex triple summation,

$$\begin{aligned} &2\pi \int_{-1}^{1} \int_0^{a} Q(r,\cos\theta) r^2 \, \mathrm{d}r \, \mathrm{d}\cos\theta \\ &= \frac{\pi\mu\omega(I\sin\alpha)^2}{8a} \sum_{l=0}^{\infty} (2l+1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{lmn} \frac{2m+1}{m(m+1)} P_m^1(\cos\alpha) \left(\frac{a}{r_0}\right)^m \\ &\quad \mathrm{Re}\left\{ \left[\frac{2n+1}{n(n+1)} \left(\frac{a}{r_0}\right)^n P_n^1(\cos\alpha) + 2\varepsilon k^* a E_n^* \right] \int_0^r R_{mn}(r) r \, \mathrm{d}r \right\} \int_{-1}^{1} P_l(x) \, \mathrm{d}x \end{aligned} \tag{60}$$

The integrals can be evaluated analytically. First, we simplify the triple summation to a single summation by making use of the following orthogonality property of Legendre functions,

$$\int_{-1}^{1} P_l(x) \, \mathrm{d}x = \begin{cases} 0; & l > 0 \\ 2; & l = 0 \end{cases} . \tag{61}$$

Using this,we may further simplify the coefficient P_{lmn} as follows:

$$\begin{aligned} P_{(l=0)mn} &= \int_{-1}^{1} P_m^1(x) P_n^1(x) \, \mathrm{d}x \\ &= \frac{1}{2} \left(m(m+1) + n(n+1) \right) \int_{-1}^{1} P_m(x) P_n(x) \, \mathrm{d}x \\ &= \begin{cases} 0; & m \neq n \\ \dfrac{2n(n+1)}{2n+1}; & m = n \end{cases} . \end{aligned} \tag{62}$$

We can now evaluate the integral involving $R_{mn}(r)$ analytically by noticing [24] that

$$\int_0^a rI_{n+1/2}(kr)I_{n-1/2}(k^*r)\,\mathrm{d}r = \frac{kaI_{n+1/2}(ka)I_{n-1/2}(k^*a) + k^*aI_{n+1/2}(k^*a)I_{n-1/2}(ka)}{2\omega\mu\sigma}\,. \tag{63}$$

With these ingredients, the two integrals in Equation (60) can be evaluated,

$$\begin{aligned} 2\pi \int_{-1}^{1}\int_0^a Q(r,\cos\theta)r^2\,\mathrm{d}r\,\mathrm{d}\cos\theta &= \frac{\pi(I\sin\alpha)^2}{2a\sigma}\sum_{n=1}^{\infty} P_n^1(\cos\alpha)\left(\frac{a}{r_0}\right)^n \mathrm{Re}\left[\frac{kaI_{n+1/2}(ka)}{I_{n-1/2}(ka)}\right] \\ &\quad \left[\frac{2n+1}{n(n+1)}\left(\frac{a}{r_0}\right)^n P_n^1(\cos\alpha) + 2\varepsilon\,\mathrm{Re}(kaE_n)\right]. \end{aligned} \tag{64}$$

The single integral term in Equation (59) can also be integrated analytically. With J_ϕ^0 substituted, we can write the integral explicitly as

$$\begin{aligned} \varepsilon 2\pi a^3 \int_{-1}^{1} P_2(\cos\theta)Q^0(a,\cos\theta)\,\mathrm{d}\cos\theta &= \frac{\varepsilon\pi\mu\omega(I\sin\alpha)^2 a}{8}\sum_{l=0}^{\infty}(2l+1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2m+1}{m(m+1)}\,\frac{2n+1}{n(n+1)}P_{lmn} \\ &\quad \times\left(\frac{a}{r_0}\right)^{m+n}\mathrm{Re}[R_{mn}(a)]P_m^1(\cos\alpha)P_n^1(\cos\alpha)\int_{-1}^{1}P_2(\xi)P_l(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{65}$$

From the orthogonality property of Legendre functions, only the term of $l = 2$ survives in the summation of l. The double integral can be simplified further if we make use of the following property of the coefficient P_{lmn}

$$\begin{aligned} P_{2mn} &= \frac{1}{2}(-6 + m(m+1) + n(n+1))\int_{-1}^{1}P_2(\xi)P_m(\xi)P_n(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2}(-6 + m(m+1) + n(n+1))\left\{\frac{3(m+1)(m+2)}{2(2m+1)(2m+3)}\int_{-1}^{1}P_{m+2}(\xi)P_n(\xi)\,\mathrm{d}\xi\right. \\ &\quad + \frac{m(m+1)}{(2m-1)(2m+3)}\int_{-1}^{1}P_m(\xi)P_n(\xi)\,\mathrm{d}\xi \\ &\quad \left. + \frac{3m(m-1)}{2(2m+1)(2m-1)}\int_{-1}^{1}P_{m-2}(\xi)P_n(\xi)\,\mathrm{d}\xi\right\}. \end{aligned} \tag{66}$$

Thus, we evaluate the single integral term analytically with the following result,

$$\varepsilon 2\pi a^3 \int_{-1}^{1} P_2(\cos\theta)Q^0(a,\cos\theta)\,\mathrm{d}\cos\theta = \frac{\varepsilon\pi(I\sin\alpha)^2}{2\sigma a}\,kak^*a\sum_{n=1}^{\infty}P_n^1(\cos\alpha)\left(\frac{a}{r_0}\right)^n H_n\,, \tag{67}$$

where

$$\begin{aligned} H_n \;=\; & \frac{3}{2(2n+3)}\,\mathrm{Re}[R_{(n+2)n}(a)]P^1_{n+2}(\cos\alpha)\left(\frac{a}{r_0}\right)^{n+2} \\ & +\frac{3}{2(2n-1)}\,\mathrm{Re}[R_{(n-2)n}(a)]P^1_{n-2}(\cos\alpha)\left(\frac{a}{r_0}\right)^{n-2} \\ & +\frac{(2n+1)(n^2+n-3)}{n(n+1)(2n+3)(2n-1)}\,\mathrm{Re}[R_{nn}(a)]P^1_n(\cos\alpha)\left(\frac{a}{r_0}\right)^{n} . \end{aligned} \tag{68}$$

The combination of Equations (64) and (68) gives the final expression for the power absorption in the deformed sphere immersed in the magnetic field generated by a single coil,

$$\begin{aligned} Q_{tot} \;=\; & \frac{\pi(I\sin\alpha)^2}{2a\sigma}\sum_{n=1}^{\infty} P^1_n(\cos\alpha)\left(\frac{a}{r_0}\right)^n \left\{\frac{2n+1}{n(n+1)}\left(\frac{a}{r_0}\right)^n\right. \\ & \quad P^1_n(\cos\alpha)\,\mathrm{Re}\left[\frac{kaI_{n+1/2}(ka)}{I_{n-1/2}(ka)}\right] \\ & +\varepsilon\left.\left(2\,\mathrm{Re}(kaE_n)\,\mathrm{Re}\left[\frac{kaI_{n+1/2}(ka)}{I_{n-1/2}(ka)}\right]+kak^*aH_n\right)\right\} . \end{aligned} \tag{69}$$

Obviously, with ε set to zero, the above equation reduces to the total power absorption for a sphere of radius a, as has been obtained previously [11].

(b) Asymptotic behavior

One of the advantages of the application of analytical methods is that the solutions may be manipulated to examine the asymptotic behavior of the system under consideration. For levitation of metals under normal conditions, the system parameter k is large. In the limit of $k \to \infty$, the Bessel function behaves, asymptotically, as follows: $I_{n+1/2}(x) \to e^x/\sqrt{z}$. With this, the current density may be shown to be concentrated near the surface region and behave exponentially decaying inward

$$J_\phi \to e^{-(a-r)/\delta}\,, \tag{70}$$

where $\delta = \sqrt{2/\mu\omega\sigma}$ is the skin depth. This is as expected from the general theory of electrodynamics [14].

With the same limit, we can show that the total power absorbed by the aspherical sample is estimated by

$$Q_{tot} \propto I^2(b_0 + b_1\varepsilon a\,\mathrm{Re}(k))\sqrt{\frac{\mu\omega}{\sigma}}\,, \tag{71}$$

where b_o and b are two constants independent of ε. If the aspherical deformation is sufficiently small or $|\varepsilon ak| \ll 1$, the major contribution will come from the zero-order term. Clearly, the total Joule-heat absorption is proportional to the square of the applied current. Its dependency on other parameters can be a complex function, unless $|\varepsilon ak| \ll 1$.

The asymptotic form of the temperature solution will be difficult to assess, except for an obvious statement resulting from Equation (56) that the local temperature is proportional to the square of the applied current and inversely proportional to the thermal conductivity.

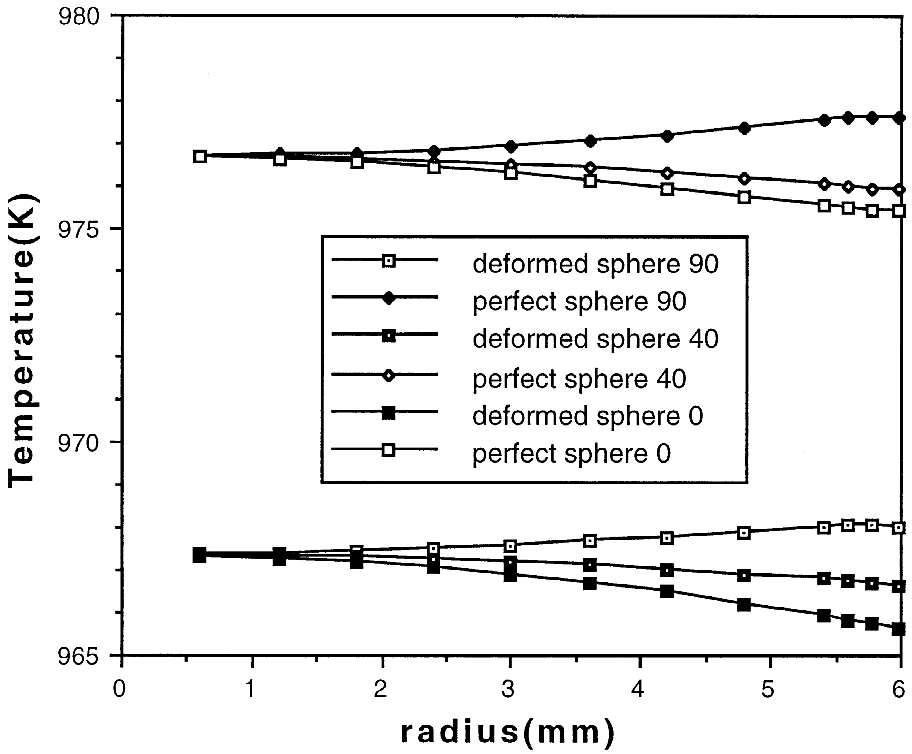

Figure 2. Comparison of radial temperature distributions in perfect and deformed spheres. The numbers in the legend refer to the θ angle.

However, a thermal balance involving only the averaged temperature in the deformed sphere will show an asymptotic behavior very similar to the total heat absorption, as $k \to \infty$.

(c) Some numerical results

The formulae derived in the previous section can also be used to obtain a detailed description of the temperature field in an aspherical sample. The special functions involved in the solution can be calculated from the expressions given by Abramowitz and Stegun [20]. While it is a very complicated and lengthy series expression, the solution actually converges rapidly, and five or six terms, more specifically, $l, m, n = 1, 2, \ldots, 5$, will normally suffice to give an accuracy of 10^{-5}. Detailed calculations also show that the first term contributes most significantly, and the contribution of other terms diminish rather quickly. For the results presented below, six terms were used and the results are accurate up to 5 digits after the decimal point. These calculations were done for a liquid aluminum drop (a=6mm) immersed in a single exciting coil located at the equator plane at a frequency of 1.45×10^{-5}Hz and $I = 212$A. Other thermophysical data for the calculations are the same as those we used earlier [11]. This coil design has been considered for magnetic levitation experiments in microgravity [10].

Figure 2 shows the calculated results for the temperature distribution in a deformed sample along the r-direction, but cut at different θ-angles. Apparently, the temperature distribution along the radius changes as a function of θ. At $\theta = 90°$, the temperature at the center is at a minimum and increases as the sample surface is approached. There exists a maximum temperature close to but not on the surface, which is attributed to the requirement that the Newtonian cooling law must be satisfied. The temperature distribution along the radius at

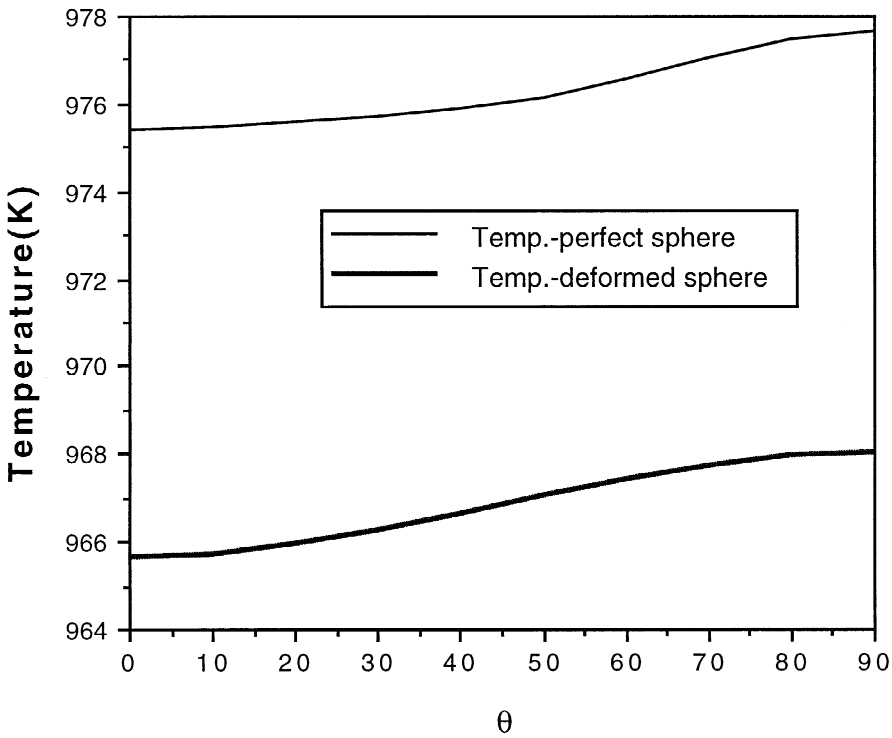

Figure 3. Comparison of temperature distributions on the surfaces of perfect and deformed spheres. Because of symmetry only the upper half is plotted.

$\theta = 0^0$ looks different, in that the temperature decays monotonically from the center to the surface of the sample. The thermal behavior of the sample along the radius is more or less intermediate between that of these two extremes when the angle changes from 0 to 90°, as examplified by the results indicated by $\theta = 40°$ also shown in the figure. As expected, this temperature distribution is such that strong Joule heating will exist near the equator, where the surface is nearest to the coil and heating decreases very quickly, both in the inward radial direction and away from the equator plane. The general behavior is very similar to that for a perfect sphere, except that the magnitude is different [11]. This decrease in Joule heating will result in a lower temperature in regions not close to the surface.

The temperature distribution on the surface of the deformed sample is compared in Figure 3 for perfect and deformed spheres. It is seen that the detailed temperature distribution along the surface is very similar, but the magnitude is different. In fact, as the sphere deforms away from the coil by 1% (i.e. $\varepsilon = 0.01$), the temperature decreases by about 10 degrees, in comparison with that for a perfect sphere. It is noted that for the results shown here, the magnitude of the temperature drop is proportional to the asphericity parameter, ε. Additional numerical computations were also made and results showed that for a 5% deformation, or $\varepsilon = 0.05$, the maximum temperature drops by about 7% [21]; however, the general distribution of the temperature distribution is almost the same as for a perfect sphere. These calculations seem to suggest that the temperature difference resulting from asphericity is approximately determined by ε when the deformation is small.

(d) Extension to multiple coils

In practice, a levitation system uses many coils arranged such that a stable levitation condition can be maintained. The solution presented above for a single-coil system can be readily extended to obtain the solutions for multiple-coil levitation configurations. Because of the linearity of the Maxwell equations, the eddy-current density and the magnetic field, within the deformed sphere induced by a coil configuration consisting of N induction coils with the same frequency and currents, can be obtained by combining the contributions from each coil,

$$J_\phi = J_{\phi 1} + J_{\phi 2} + J_{\phi 3} + \cdots + J_{\phi N}. \tag{72}$$

The time-averaged power absorption by the aspherical sample in a levitation system consisting of N current loops can then be calculated by the same procedure as we described above

$$\begin{aligned} Q_{tot} \;=\; & \frac{\pi I^2}{2a\sigma} \sum_{i=1}^{N} \sum_{j=1}^{N} \sin\alpha_i \sin\alpha_j \sum_{n=1}^{\infty} P_n^1(\cos\alpha_i) \left(\frac{a}{r_{0i}}\right)^n \\ & \times \left\{ \frac{2n+1}{n(n+1)} \left(\frac{a}{r_{0j}}\right)^n P_n^1(\cos\alpha_j)\, \mathrm{Re}\left[\frac{kaI_{n+1/2}(ka)}{I_{n-1/2}(ka)}\right] \right. \\ & \left. + \varepsilon \left(2\, \mathrm{Re}(kaE_{n,j})\, \mathrm{Re}\left[\frac{kaI_{n+1/2}(ka)}{I_{n-1/2}(ka)}\right] + kak^* aH_{n,j} \right) \right\} , \end{aligned} \tag{73}$$

where the subscript j on $H_{n,j}$ means that r_{0j} and α_j should be used to compute Q_{tot}.

The temperature solution can also be obtained. The procedure should be straightforward. For the same multiple-coil configuration considered, we can show that the temperature distribution with a linear accuracy is given by

$$\begin{aligned} T(r,\theta) \;=\; & \frac{\mu\omega I^2}{16K} \sum_{l=0}^{\infty} P_l(\cos\theta) \sum_{i=1}^{N} \sum_{j=1}^{N} \sin\alpha_i \sin\alpha_j \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{2m+1}{m(m+1)} \frac{2n+1}{n(n+1)} \right. \\ & \times \left(\frac{a}{r_0}\right)^{n+m} P_m^1(\cos\alpha_i) P_n^1(\cos\alpha_j) \left(I_{lmn}(r) P_{lmn} - \frac{g_{lmn}}{l+ha} \right) + \\ & \left[\int_0^r \left(\frac{r'}{r}\right)^{l+1} F_{lmnij}(r')\, \mathrm{d}r' + \int_r^a \left(\frac{r}{r'}\right)^l F_{lmnij}(r')\, \mathrm{d}r \right] \\ & \left. + \left(\frac{r}{a}\right)^l \left[\frac{(l+1-ha)}{l+ha} \int_0^a \left(\frac{r'}{a}\right)^{l+1} F_{lmnij}(r')\, \mathrm{d}r' \right] \right\} , \end{aligned} \tag{74}$$

where $F_{lmnij}(r)$ is a function of α_i and α_j and is calculated by

$$F_{lmnij}(r) = P_{lmn} P_m^1(\cos\alpha_i) \frac{4(2m+1)}{m(m+1)} \left(\frac{a}{r_0}\right)^m \mathrm{Re}[k^* E_{n,j}^* R_{mn}(r)] . \tag{75}$$

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